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Possible representations of semisimple groups $SU(m_1)\times SU(m_2)\times\ldots\times SU(m_k)$ for finite $N=2$ supersymmetric Yang–Mills theories

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Abstract. In this paper we list all possible representations of the semisimple groups $SU(m_1)\times SU(m_2)\times\ldots\times SU(m_k)$ for finite $N=2$ supersymmetric Yang-Mills theories.

Recently a lot of attention has been paid to the finite quantum field theories. The first such example is $N=4$ supersymmetric Yang–Mills (SYM) theory (Gliozzi *et al* 1977, Brink *et al* 1977), which was first proved to be finite in the light-cone gauge formalism (Mandelstam 1983, Brink *et al* 1983). The more general cases of finite quantum field theories are $N=2$ SYM theories with $N=2$ matter multiplets provided their gauge coupling β function vanishes at one loop (Howe *et al* 1983). The $N=2$ SYM theories include an $N=2$ vector multiplet which consists of an $N=1$ vector multiplet and an $N=1$ chiral scalar multiplet all in the adjoint representation of gauge group G, and several $N=2$ matter multiplets each of which consists of an $N=1$ chiral scalar multiplet in representation R_i and an $N=1$ chiral scalar multiplet in representation $\bar{R}_i$. The vanishing of the β function at one-loop level for simple group G becomes

$$C_2(\mathrm{G})=\sum_i T(R_i) \tag{1}$$

where $C_2(\mathrm{G})$ is the value of the quadratic Casimir operator for the adjoint representation of G, and $T(R_i)$ is the Dynkin index of the representation R_i. Representations of all classical simple groups satisfying (1) are found (Koh and Rajpoot 1984) and the candidates for finite grand unified theories which can accommodate at least three generations of ordinary quarks and leptons, are discussed (Dong *et al* 1984). When G is a semisimple group $\mathrm{G}=\mathrm{G}_1\times\mathrm{G}_2\times\ldots\times\mathrm{G}_k$ where $\mathrm{G}_1,\ldots,\mathrm{G}_k$ are simple groups, criterion (1) for finite $N=2$ SYM theories can be easily extended to

$$\begin{aligned} C_2(\mathrm{G}_1)&=\sum_i T(R_1^{(i)})\times\dim R_2^{(i)}\times\ldots\times\dim R_k^{(i)}\\ C_2(\mathrm{G}_2)&=\sum_i \dim R_1^{(i)}\times T(R_2^{(i)})\times\ldots\times\dim R_k^{(i)}\\ &\vdots\\ C_2(\mathrm{G}_k)&=\sum_i \dim R_1^{(i)}\times\dim R_2^{(i)}\times\ldots\times T(R_k^{(i)}) \end{aligned} \tag{2}$$

where $N=2$ matter multiplets are in the representations $R^{(i)}=R_1^{(i)}\times R_2^{(i)}\times\ldots\times R_k^{(i)}$ and dim R is the dimension of representation R. We have found representations of all semisimple classical groups $G=G_1\times G_2$ (Jiang and Zhou 1984) which may be relevant to preon models. In this paper we will find all representations of semisimple groups $G=SU(m_1)\times SU(m_2)\times\ldots\times SU(m_k)$ which satisfy equation (2). We have worked out the $SU(m_1)\times SU(m_2)$ case (Jiang and Zhou 1984). So we only discuss the cases of $K\geqslant 3$ here. If equations (2) can be decomposed into more than one set of equations which are independent of each other, we call the case reducible. Every reducible case can be reduced into several irreducible cases where equations (2) are dependent on each other. We need only discuss irreducible cases. Suppose there is a set of representations $R^{(i)}$ of G satisfying (2). In an irreducible case for each subgroup $SU(m_j)$ of G there must exist a representation $R^{(i)}$ in the set which is not a singlet for $SU(m_j)$ as well as at least for another subgroup $SU(m_1)$, i.e. $R_j^{(i)}$ and $R_1^{(i)}$ are not singlets. When the number k of subgroups is greater than two, $R_j^{(i)}$ and $R_1^{(i)}$ can only be fundamental representations of $SU(m_j)$ and $SU(m_1)$ respectively. There is only one case where a representation of G is not a singlet for more than two groups. It is the type A in table 1, where $G=SU(2)\times SU(2)\times SU(2)$ and only one representation $R=\square\times\square\times\square$ of G satisfying (2). $\square$ denotes the fundamental representation of $SU(m)$. In all other cases every representation $R^{(i)}$ in a set satisfying (2) is not a singlet at most for two subgroups of G, and the representations $R_j^{(i)}$ and $R_1^{(i)}$ of the two subgroups must be fundamental representations.

Now let us use a diagram to denote a set of representations of G satisfying (2). Each subgroup $SU(m_i)$ of G is denoted by a dot in the diagram and the value m_i is written near the dot to express the subgroup is $SU(m_i)$. If there is a representation $R^{(i)}$, such that $R_j^{(i)}$ and $R_1^{(i)}$ are fundamental representations for subgroups $SU(m_j)$ and $SU(m_1)$, we connect the two dots for $SU(m_j)$ and $SU(m_1)$ by a line. There are still representations in the set which are non-singlets only for one subgroup. Such representations can only be three-rank antisymmetric representation $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ of $SU(m)$ $(m=6)$, two-rank antisymmetric representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of $SU(m)(m\geqslant 4)$, two-rank symmetric representation $\square\square$ of $SU(m)$ $(m\geqslant 3)$ and fundamental representation $\square$ of $SU(m)$ $(m\geqslant 2)$ (Koh and Rajpoot 1984). We directly write down these representations near the dots of the corresponding subgroups $SU(m_i)$. So in such a diagram all representations in a set satisfying (2) are expressed clearly. If a set of representations is irreducible, its diagram is a connected one where there is no subset of dots which is not connected with other dots in the set. Type B in table 1 is a ring where all values of the dots on the ring are equal and no representation which is a non-singlet only for one subgroup can be added in. Furthermore, we cannot add any branch to the ring. Therefore the remaining types of diagrams all take tree shapes.

A dot in a diagram with n branches is called n-branch dot. It is easy to verify that the only diagram with a four-branch dot is type C in table 1. A dot which has more than four branches cannot appear in a diagram. If there are two three-branch dots in a diagram, it must be type D in table 1. There is no diagram with more than two three-branch dots. There are only two types of diagram left: one-branch type E (chain) and three-branch type F with only one three-branch dot.

It is easy to see that if the value of a dot is greater than the value of the next dot on a chain, the values of dots on the chain afterwards must decrease dot by dot. Therefore chains can be divided into four kinds E1, E2, E3 and E4 listed in figure 1 according to the change trend of the values on the chains. E1 is a flat type where all values on a chain are equal. E2 is a flat-slant one where the values on one part of a chain are equal, and those on another part decrease. E3 is a slant-flat-slant one where

Table 1. All possible irreducible diagrams satisfying equations (2)†.

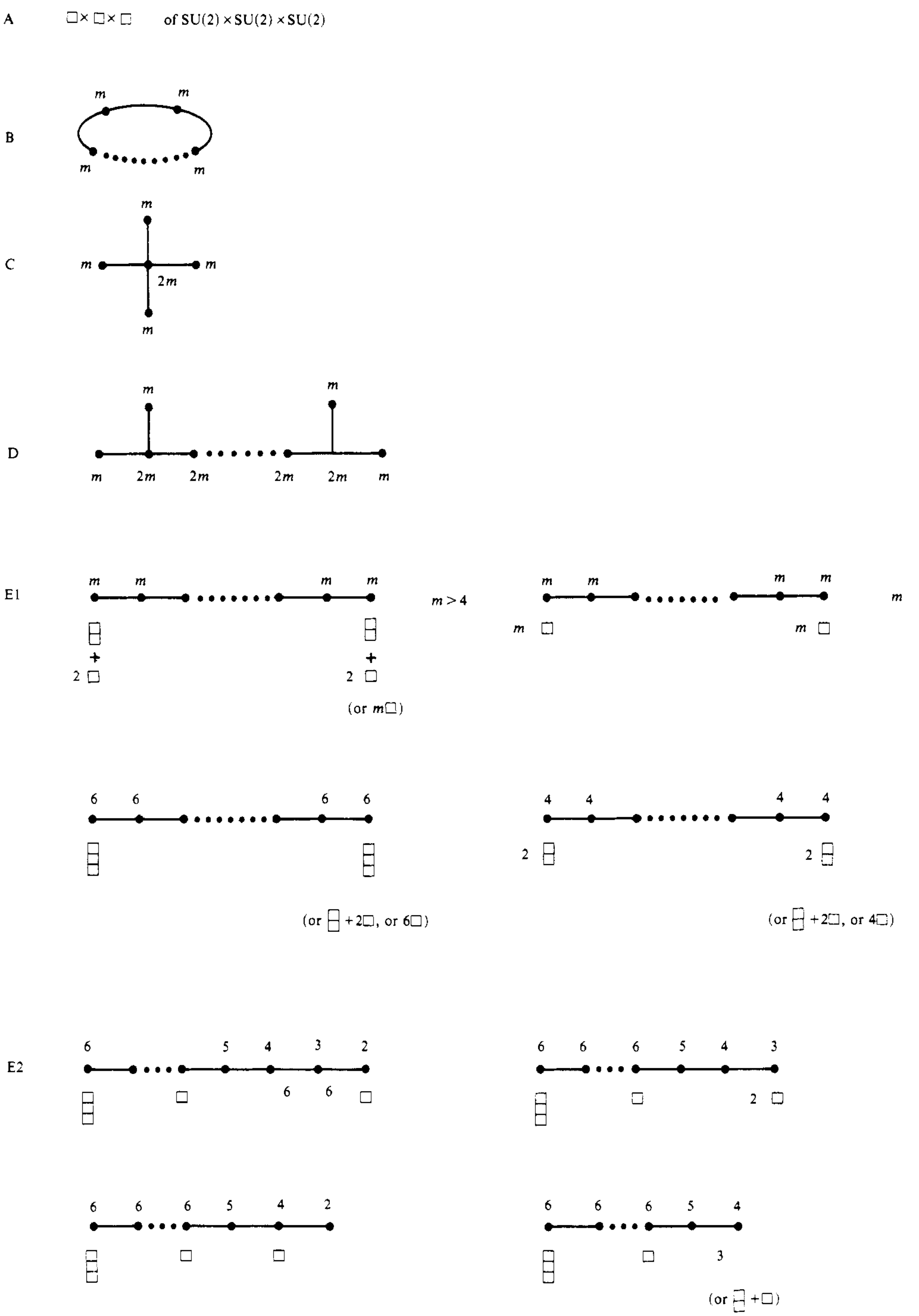


Table 1. (continued)

E3

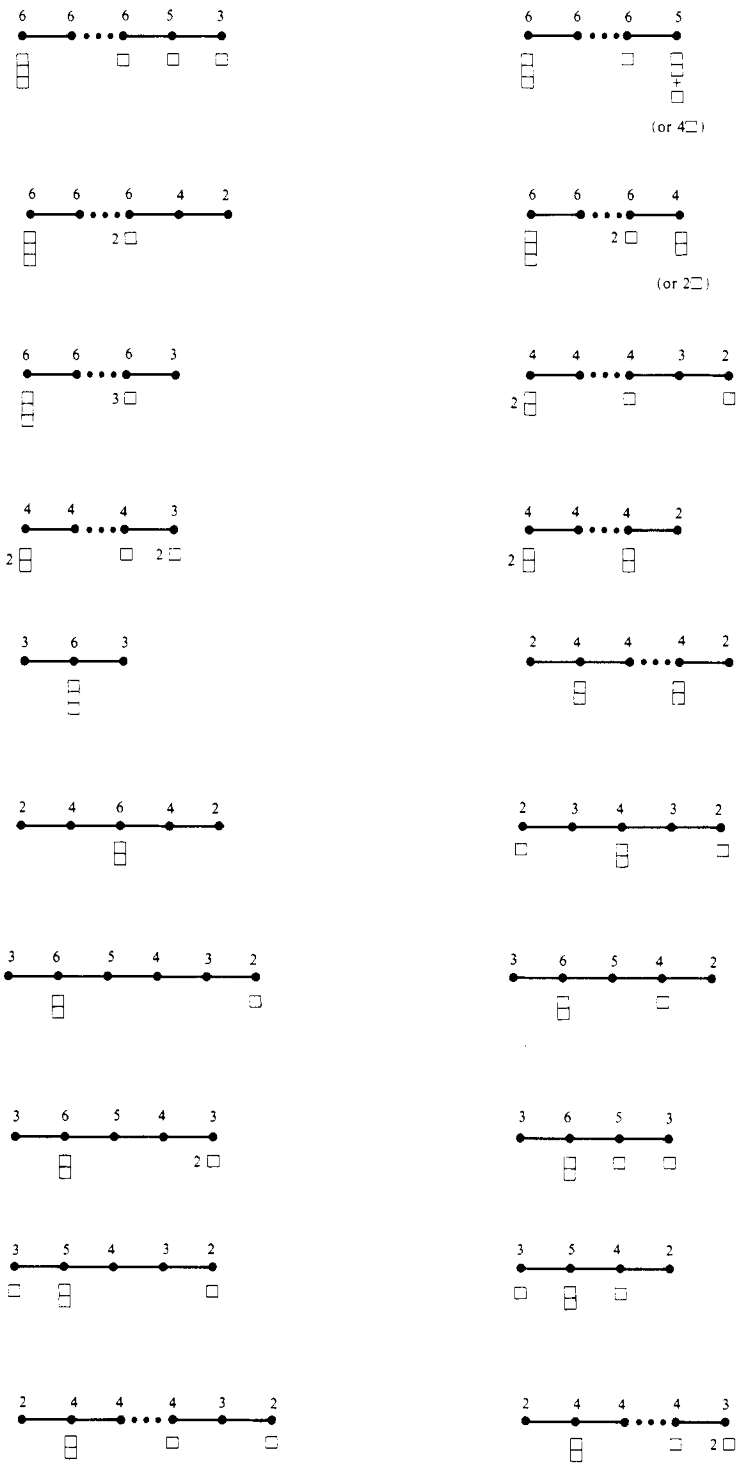


Table 1. (continued)

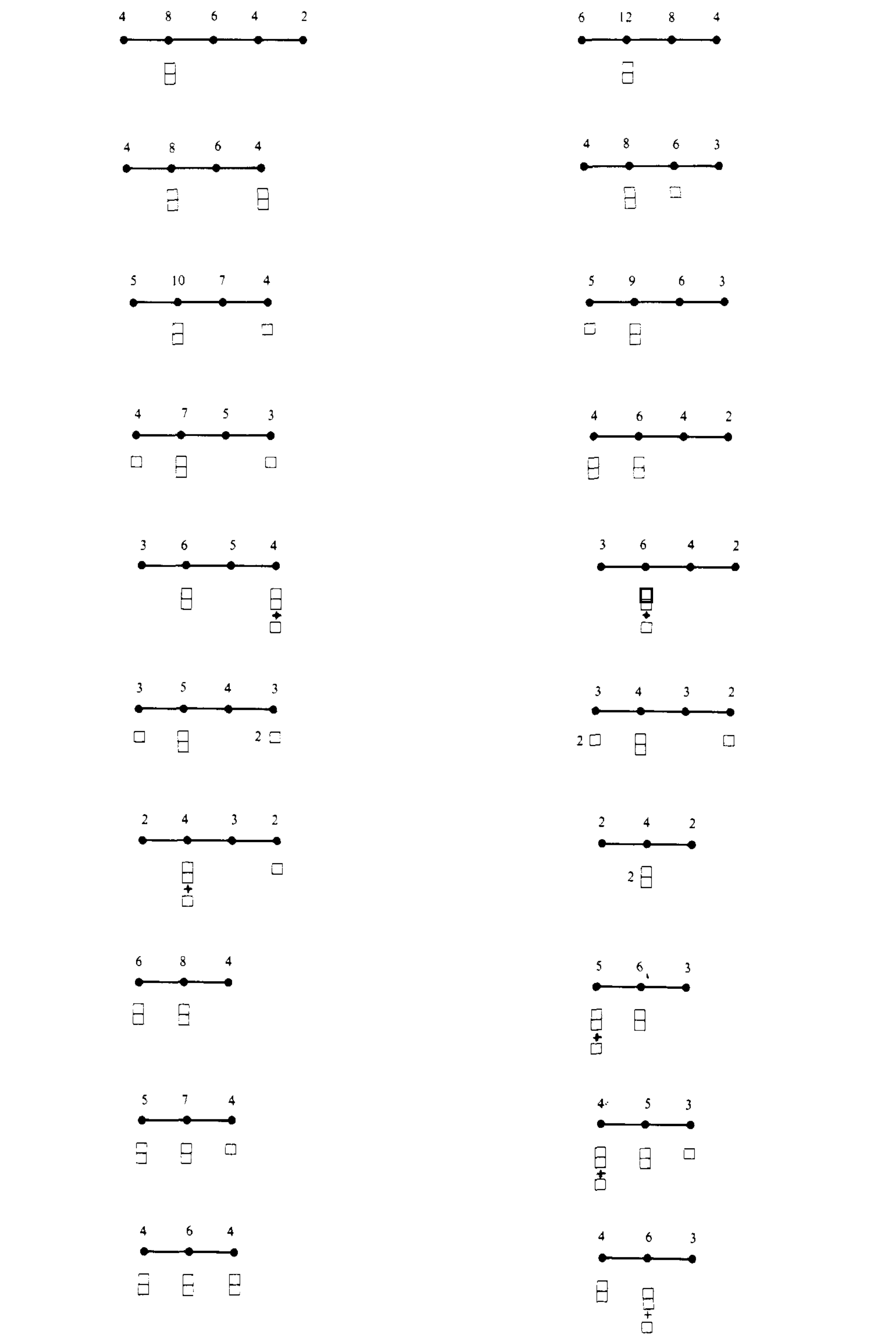


Table 1. (continued)

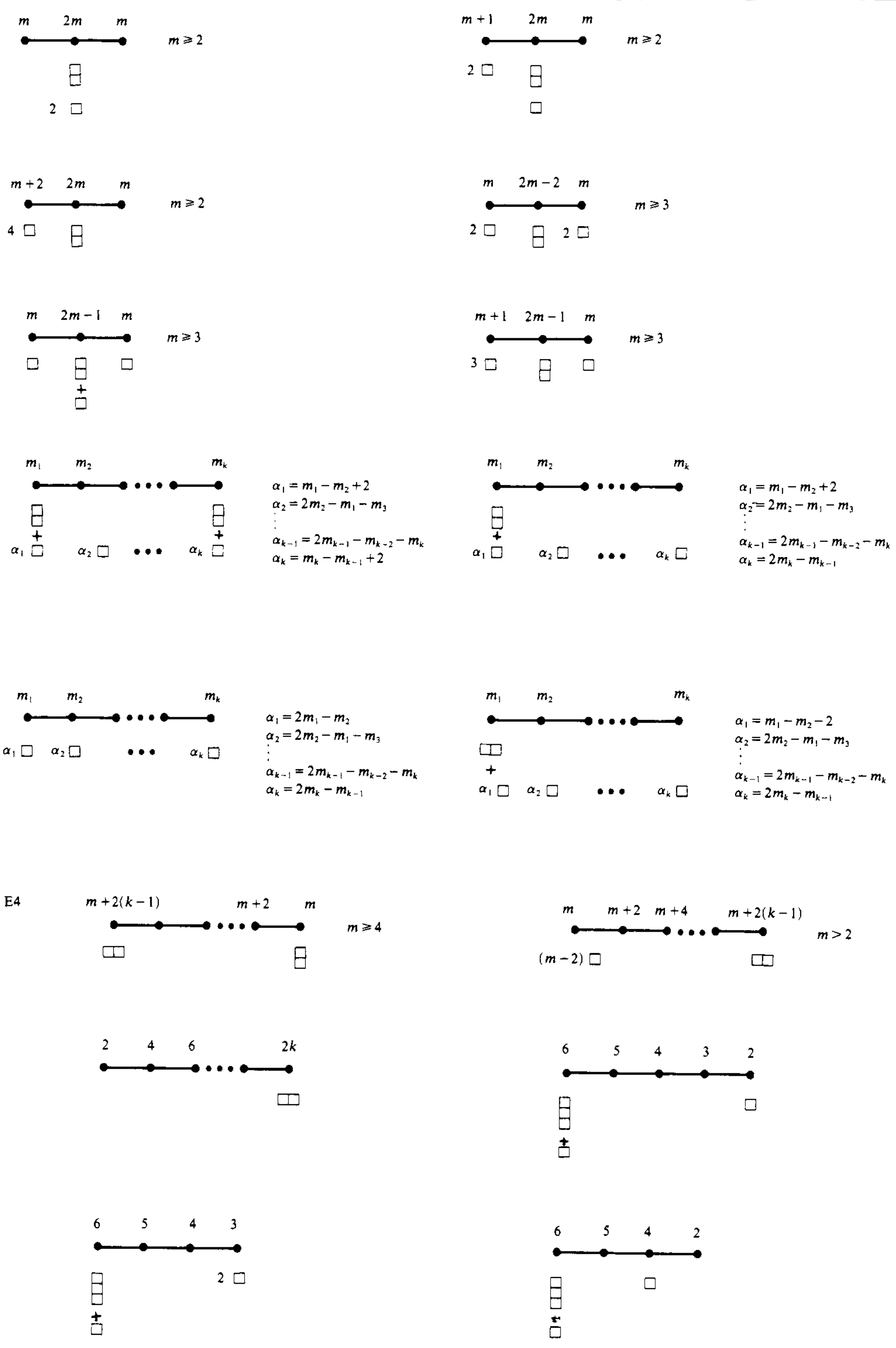
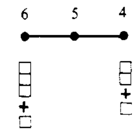
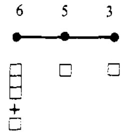
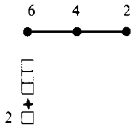
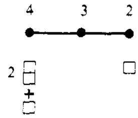


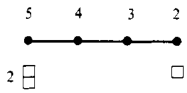
Table 1. (continued)

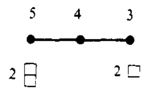


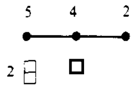


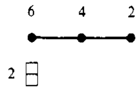




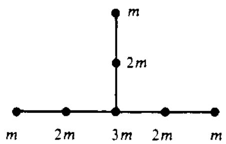


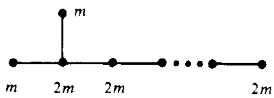




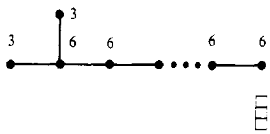


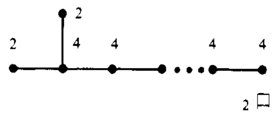
F1

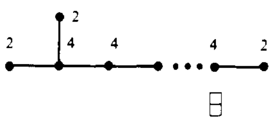




F2







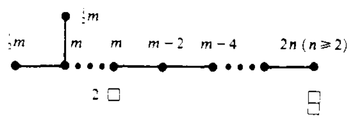
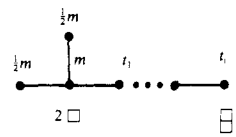
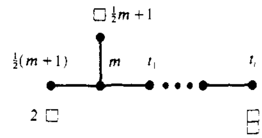
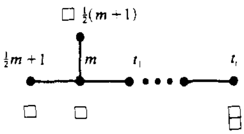
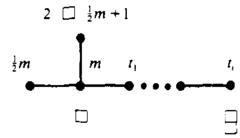
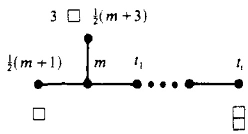
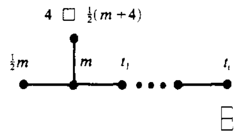
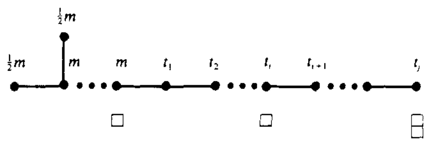
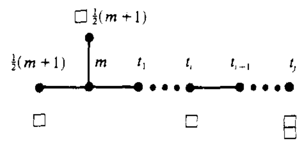
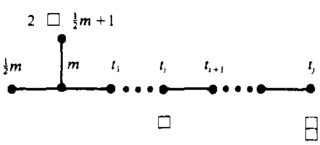


Table 1. (continued)

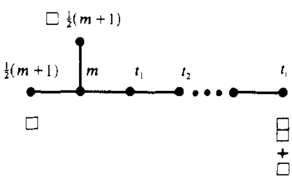
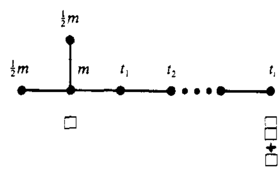
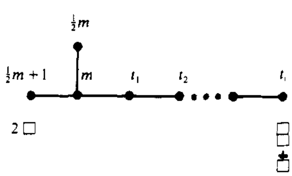
$m - t_1 = t_1 - t_2 = \ldots = t_{i-1} - t_i = 2 \qquad t_i \geqslant 4$

F3

$m - t_1 = t_1 - t_2 = \ldots = t_{i-1} - t_i = 1$
$t_i - t_{i+1} = \ldots = t_{j-1} - t_j = 2, \qquad t_j \geqslant 4$

F4

$m - t_1 = t_1 - t_2 = \ldots = t_{i-1} - t_i = 1 \qquad t_i \geqslant 4$

Table 1. (continued)

F5

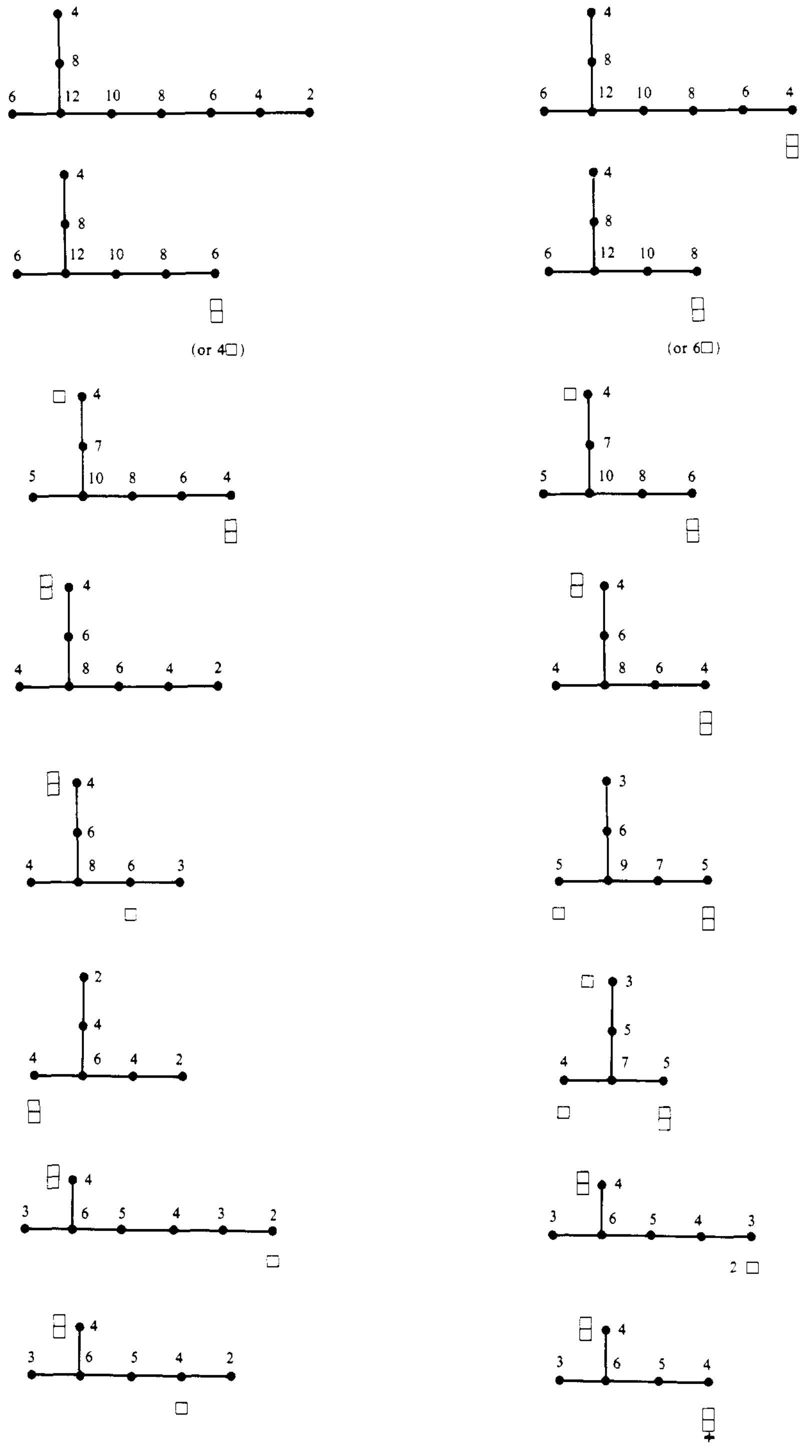


Table 1. (continued)

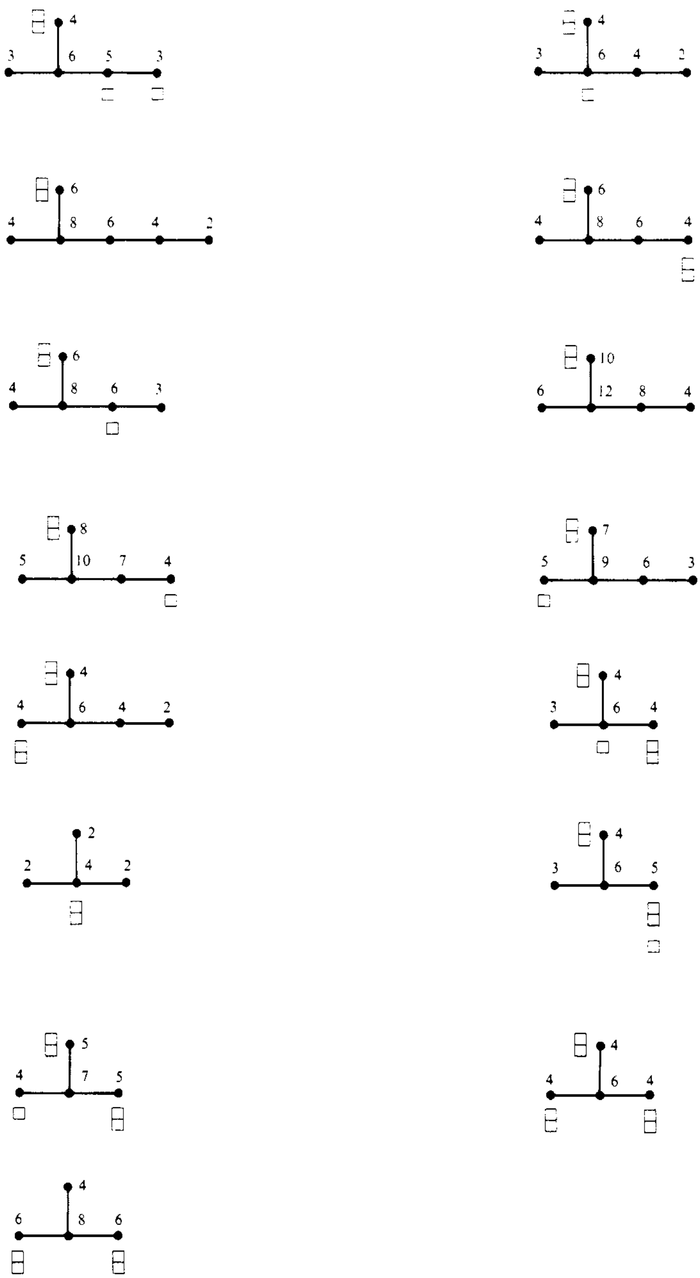
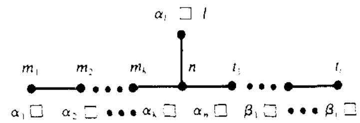


Table 1. (continued)

F6		$\alpha_1 = 2m_1 - m_2$ $\alpha_2 = 2m_2 - m_1 - m_3$ $\vdots$ $\alpha_k = 2m_k - m_{k-1} - n$ $\alpha_n = 2n - m_k - l - t_1$ $\alpha_l = 2l - n$	$\beta_1 = 2t_1 - n - t_2$ $\beta_2 = 2t_2 - t_1 - t_3$ $\vdots$ $\beta_{i-1} = 2t_{i-1} - t_{i-2} - t_i$ $\beta_i = 2t_i - t_{i-1}$

† If the dots with value 6 have $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, they always may have $\begin{smallmatrix}\square\\\square\end{smallmatrix}+2\square$, or $6\square$. For simplicity sometime we only list $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ in diagrams. Similarly if the dots with value 4 have $2\begin{smallmatrix}\square\\\square\end{smallmatrix}$, they always may have $\begin{smallmatrix}\square\\\square\end{smallmatrix}+2\square$, or $4\square$ and also for brevity sometimes we only list $2\begin{smallmatrix}\square\\\square\end{smallmatrix}$. In the same spirit when we see dots with value m have $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, instead $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ they may have $(m-2)\square$.

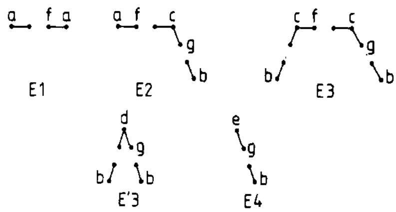


Figure 1.

some dots inside a chain have the same values and values on the two wings decrease monotonically. The extreme case of E3 is E′3 where no dots inside the chain have the same values. E4 is a slant one where values on a chain decrease from one end to another end.

In order to solve equation (2) in the chain case we have to know what representations which are non-singlet only for one subgroup, can be added to the dots in a chain. Let us discuss the question in some detail. There are several kinds of dots in figure 1. We cannot add any representations on those dots which are inside a flat part of a chain, such as dots f in figure 1. We can only add fundamental representations on the dots inside a slant part of a chain, such as the dots g in figure 1. The end dots of a flat part in a chain, such as dots a in figure 1, can only have $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ $(m=6)$, or $2\begin{smallmatrix}\square\\\square\end{smallmatrix}$ $(m=4)$, or $(\begin{smallmatrix}\square\\\square\end{smallmatrix}+2\square)$ $(m\geq 4)$, or $m\,\square$ where m is the value of the end dots. The turning dots between a flat part and a slant part, such as dots c in figure 1, can only have $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ $(m=4)$ or $\square$ where m is also the value of the turning dots. The end dots of a slant part at the low end, such as b in figure 1, can have representations $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square$. The turning dot d in figure 1 can have $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square$. The end dots e of a slant part at the upper end may have any allowed representations, $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ $(m=6)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, $\square\square$, and $\square$. All possible diagrams of chains are listed in type E in table 1.

The last type F in table 1 is the three-branch diagrams. The lengths of the three branches cannot be arbitrary long. Type F1 is a typical example where each branch has two dots (not including the three-branch dot) and we cannot make any branch longer in type F1.

All diagrams for $G = SU(m_1)\times SU(m_2)\times\ldots\times SU(m_k)$ $(k\geq 3)$ which satisfy equations (2) are listed in table 1. We can give an example to show how to use the table. For example, the first diagram of type E3 in table 1 means gauge group $G = SU(3) \times SU(6) \times SU(3)$ and the representations $R^{(i)}$ of $N = 2$ matter multiplets are

$$\square \times \square \times . + . \times \square \times \square + . \times \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \times . \qquad \text{(. denotes singlet)}$$

where in addition we have to include all representations $\bar{R}^{(i)}$ of $N = 2$ matter multiplets which are not shown in the diagram. All $N = 2$ matter multiplets of $R^{(i)}$ and $\bar{R}^{(i)}$ and the $N = 2$ vector multiplet in the adjoint representation of G together construct a finite $N = 2$ SYM theory. At present the cases of $SU(m_1) \times SU(m_2) \times SU(m_3)$ may be relevant to preon models (for example see Lyons 1982). For convenience in table 2 we list Dynkin indexes and dimensions of those representations which are needed in our calculations.

Table 2. The dimension and Dynkin index of representations of SU(m).

Representation R_i	Dimension	$T(R_i)$	Range of m
$\square$	m	1/2	$m \geqslant 2$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\frac{1}{2}m(m-1)$	$\frac{1}{2}(m-2)$	$m \geqslant 4$
$\square\square$	$\frac{1}{2}m(m+1)$	$\frac{1}{2}(m+2)$	$m \geqslant 3$
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	$\frac{1}{6}m(m-1)(m-2)$	$\frac{1}{4}(m-2)(m-3)$	$8 \geqslant m \geqslant 6$
$m-1\left\{\begin{smallmatrix}\square\square\\\square\\\vdots\\\square\end{smallmatrix}\right.$	m^2-1	m	$m \geqslant 2$

Acknowledgments

We would like to thank T S Tu for beneficial discussions.

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